Resistance in cell membrane and nerve fiber

Ji-Huan He a,b, *

a College of Science, Donghua University, Yan’an Xi Lu Road, Shanghai 200051, China
b Key Lab of Textile Technology, Ministry of Education, China

Received 4 August 2004; accepted 25 September 2004

Abstract

A mathematical model describing the resistance in cell membrane and nerve fiber is proposed, which is naturally different from that for metal conductors. An allometric scaling law between the resistance and the section area is obtained. In the derivation, He Chengtian’s interpolation, which has millennia history, is applied. Most famous models, such as the Hodgkin–Huxley model, FitzHugh–Nagumo models, should be, therefore, revised.

© 2004 Elsevier Ireland Ltd. All rights reserved.

Keywords: Nerve fiber; Hodgkin–Huxley model; FitzHugh–Nagumo model; Allometric scaling; He Chengtian’s interpolation

We know from Ohm’s law that current flows down a voltage gradient in proportion to the resistance in the circuit. Current is therefore expressed as:

\[ I = \frac{V}{R} = gV, \] (1)

where \( I \) is the current, \( V \) is the voltage, \( R \) is the resistance, and \( g \) is the conductance. In the contemporary electrophysiological models, the ionic permeabilities of the cell membrane act as resistors in an electronic circuit, and this conceptual idea is widely applied in cell biology, for example, in Hodgkin–Huxley model [10], FitzHugh–Nagumo model [3], Frankenhaeuser–Huxley model [4], and others.

The resistance, \( R \), in (1) is expressed in the form:

\[ R = \frac{kL}{A}, \] (2)

where \( A \) is the area of the conductor, \( L \) its length, and \( k \) resistance parameter. Eq. (2) is valid only for metal conductors where there are plenty of electrons. However, in cell membranes or in nerve fiber, the current is caused not by electrons, but by ions (Na⁺, K⁺, non-specific leak), so Eq. (2) should be modified in order to accurately describe the ionic currents. Though many experiment observations show the deviation of Eq. (1) when applied to biology, our mathematical understanding of this fundamental phenomenon is scarce and primitive, and its accurate mechanism remains ambiguous, this makes it very difficult to deduce a mathematical model describing the ambiguous mechanism.

In this letter, we will establish an allometric scaling law for the discussed problem by He Chengtian’s interpolation [5].

Hodgkin–Huxley equations [10] describe ionic currents of the squid giant axon, which were obtained by experiment, rather than theoretical derivation. These equations describe the ionic currents of an unmyelinated neuron that were derived from experiments on voltage- and space-clamped whole axons. The equations include a depolarizing Na⁺ current, a repolarizing K⁺ current and a non-specific leakage current.

The research on Hodgkin–Huxley model is carried out along two directions. One is experiment study, i.e. obtain data with advanced experimental technologies so as to improve the mathematical form of Hodgkin–Huxley model. The other is mathematical analysis on the model. Due to the character of multi-parameters, strong coupling and non-linear, non-linear theories such as chaos and bifurcation are widely used to perform the analysis [11]. Despite that these studies show...
promising indicative results, a complete theoretical analysis is rare. And resistance calculation is still based on Eq. (2), which leads to inaccuracy of Hodgkin–Huxley model. As illustrated in Fig. 1, the mechanism of ion currents is different from that of electron currents.

Later, methods were developed to voltage-clamp single nodes in myelinated fibers. The Frankenhaeuser–Huxley (FH) equations are the result of voltage-clamp studies of the nodal membrane of amphibian, myelinated neurons [4]. The basis of these equations is similar to that of the HH model, although a fourth, delayed sodium current was also described. There exist huge publications on neural models (see[1], references cited thereby), but the fundamental resistance formulation has not yet dealt with. We will apply the allometric approach [6–8,12,13] to the derivation of resistance formulation for the discussed problem.

The resistance for Ohm conductor (see Fig. 2) scales as

\[ R_c \sim \frac{1}{A} \sim r^{-2}, \tag{3} \]

where \( R_c \) is the resistance, \( r \) radius of the conductor, \( A \) its section area. So for the Ohmic bulk conduction current, we have

\[ I_c \sim R_c^{-1} \sim A \sim r^2, \tag{4} \]

which corresponds to

\[ I_c = \pi r^2 kV, \tag{5} \]

where \( k \) is the dimensionless conductivity of the conductor, \( V \) applied electric field.

The resistance for surface convection (see Fig. 3), which occurs in electrospinning [9,14] and charged flow [2], scales as

\[ R_s \sim 1 \sim r^{-1}, \tag{6} \]

as

\[ R_s \sim \frac{1}{A^{1/2}} \sim r^{-1}. \tag{7} \]

For the surface convection current, we have

\[ I_s \sim R_s^{-1} \sim A^{1/2} \sim r, \tag{7} \]

which corresponds to

\[ I_s = 2\pi r \sigma u, \tag{8} \]

where \( \sigma \) is surface density of the charge.

For an insulator, the resistance is independent of its section area, in scaling form, we write

\[ R \sim \frac{1}{A^{0.1}}. \tag{9} \]

It is known that propagation of nerve impulse can be described in terms of membrane transport driven by ion gradients, which in turn crucially depend on the position and distribution of charged particles called gating particles. There exist three ionic currents in an electrically active membrane, one for Na\(^+\), one for K\(^+\), and one for a non-specific leak. The ionic currents can neither be considered as Ohmic bulk conduction, nor surface convection, so the scaling (6) is also valid for calculation of the resistance for neuron.

For neuron resistance, we assume that

\[ R \sim \frac{1}{A^{0.5}}, \tag{10} \]

where \( \beta \) is constant relative to conductive character of cell membrane or nerve fiber. The conductive behavior of cell membrane or nerve fiber lies between that for metal and that for insulator, so the value of \( \beta \) lies between 0 and 1, i.e.

\[ 0 < \beta < 1. \tag{11} \]

In ancient China, there were many interpolation formulae, among others, we will use hereby He Chengtian’s interpolation [5] to fix approximately value of \( \beta \).

Consider the following inequality

\[ \frac{a}{b} < s < \frac{c}{d}, \tag{12} \]
Fig. 4. Resistance for non-Ohmic bulk conductor: $R \sim \frac{1}{AD^{D+1}}$, where $D$ is fractal dimension of moving charge in the section.

where $a$, $b$, $c$, and $d$ are integers. According to He Chengtian’s interpolation, the value of $x$ can be approximately identified as follows:

$$x = \frac{am + cn}{bm + dn}, \quad (13)$$

where $m$ and $n$ are weighting factors. It is easy to prove that

$$\frac{a}{b} < \frac{am + cn}{bm + dn} = \frac{c}{D}. \quad (14)$$

Applying He Chengtian’s interpolation, in view of Eq. (11), the value of $\beta$ can be written in the form

$$\beta = \frac{n}{m+n} = \frac{n/m}{n/m+1} = \frac{D}{D+1}. \quad (15)$$

where $m$ and $n$ are integers, $D = m/n$.

So for the discussed problem, the resistance scales as

$$R \sim \frac{1}{AD^{D+1}}. \quad (16)$$

Hereby $D$ can be considered as the fractal dimension of the section (see Fig. 4).

In order to verify the allometric scaling (16), we consider the surface convection current, the fractal dimension of charged section is $D = 1$, leading to (6).

To summarize, we have proposed a theoretical model dealing with for the first time a seemingly complex currents in biology. This allometric model is able to describe a complex dynamic process from the theory, and it requires less empirical or semi-empirical input. Of course the author understands that no matter how rigorous, some experimentally verification is needed to validate the model.

Acknowledgement

The work is supported by grant 10372021 from National Natural Science Foundation of China.

References